and $\log(x+iy)+\log(x'+iy')=\log[(x+iy)(x'+iy')]$. In a second paper[1] of the year 1856 he states that this last equation is not always satisfied and he investigates the cases of failure. In 1869 was published an article by Cayley, *On Logarithms of Imaginary Quantities*,[2] in which he treats of the principal or "selected" values, adopting $-\pi$ and $+\pi$ as the angular limits.

The German term for principal value is *Hauptwert*, a term used by Karl Weierstrass in his lectures on Abelian transcendents delivered about 1875 in Berlin. Cauchy had taken, as the principal value of $\sqrt{a+bi}$, the value whose second coördinate has the same sign as the second coördinate b, in $a+bi$. Weierstrass modified this definition, so that the principal value of $\sqrt{a+bi}$ is the "positive"[3] value of the square root, provided that $a+bi$ is itself "positive." Weierstrass calls $a+bi$ "positive," when a is positive or, if $a=0$, when b is positive. If $a+bi$ is negative, then the principal value of its square root shall be $i\sqrt{-a-bi}$, where $\sqrt{-a-bi}$ is "positive."

The terminology and notation of powers, roots, and logarithms received further attention from Cauchy in 1846.[4] When the variables in a function are extended from real to imaginary values, the notation and the formulas used for real variables cannot be conserved, unless one adopts new conventions to fix the meaning of the notation for imaginaries. Since a number has an infinity of logarithms, one cannot represent them all by one and the same notation without introducing confusion into the calculus. He lets lx stand for that natural logarithm of x in which the coefficient of $\sqrt{-1}$ lies between $+\pi$ and $-\pi$, the lower limit being excluded. Using Lx for the logarithm in any system, he has $Lx=Le^{lx}$, $x^a=e^{alx}$, $x^y=e^{ylx}$. With the above restriction these functions are in general *continuous* functions of x for real and complex values of the variables. These values Cauchy called the *logarithme principale, puissance principale,*—he used similarly the term *racine principale*. Part of this terminology was first suggested by E. G. Björling.

The notation $((1))^x$ was used in 1845 by T. Wittstein of Hanover in a proof that every equation has a root.[5] It was used in 1856 by the Swiss, H. Kinkelin[6] and is used by O. Stolz and J. A. Gmeiner in their well-known work, *Theoretische Arithmetik*.[7]

As a counterpart to Cauchy's notation for the general power, G. Peano has introduced the notation $\sqrt[m]{*}a$ to designate all the valu^{e}s that the mth root of a may take.[8]

Cauchy discusses the limit toward which $(1+z/m)^m$ converges, for increasing positive integral values of the variable m, when z may be a complex number.[9]

[1] Cayley, *Coll. Math. Papers*, Vol. III, p. 209.
[2] *Op. cit.*, Vol. VI, p. 14.
[3] K. Weierstrass, *Math. Werke*, Vol. IV, Berlin, 1902, pp. 348–350.
[4] *Comptes Rendus*, T. XXIII, p. 271; also *Exercices d'Analyse et d. Phys. Math.*, IV, 1847, p. 253; *Œuvres*, 1. S., T. X, p. 75.
[5] *Grunert's Archiv*, Vol. 6, 1845, p. 230.
[6] Same journal, Vol. 27, 1856, pp. 304–315.
[7] *Theor. Arith.*, 2 Abth., Leipzig, 1902, p. 373.
[8] *Formulaire de mathématiques* publié par la "Rivista di matematica," T. I, Turin, 1895, p. 19.
[9] Cauchy, *Exercices d'analyse et de phys. math.*, T. IV, Paris, 1847, pp. 232–246.

This limiting value, which is shown to exist, lends itself to the definition of the exponential function e^z. It follows that $e^z \cdot e^{z'} = e^{z+z'}$. Cauchy proceeds to the generalized power, in which both base and exponent are complex, or *quantités géométriques* as Cauchy calls them here.[1] If z and u are both complex, then z^u needs definition; he does it by the equation $z^u = e^{ul(z)}$. This definition includes, as special cases, the less general developments previously discussed by Cauchy. He states that E. G. Björling has given the same definition of z^u and has introduced the term *puissance principale*. This term Cauchy adopts, only he attributes to the argument p in $z^u = r^n e^{upi}$ values $-\pi < p \leqq \pi$, while Björling set the limits $(\pi/2) + \pi > p \geqq (\pi/2) - \pi$. By Cauchy's scheme, two conjugate complex numbers raised to powers indicated by conjugate numbers, respectively, yield for principal values, complex quantities which are themselves conjugate.

While the descartian exponential notation is excellent in all ordinary writing and printing, it is unsatisfactory for complicated exponents. Changes in notation have been proposed by De Morgan at the end of his Calculus of Functions, in the *Encyclopædia Metropolitana* (1843), and also in an article in the *Cambridge Philosophical Transactions*,[2] and later by J. W. L. Glaisher.[3] De Morgan would print "a raised to the power $(a + bx)/(c + dx)$" thus, $a \wedge \{(a + bx)/(c + dx)\}$. Glaisher likewise suggests the use of arrows which he prefers to a third notation that had occurred to him, namely, the printing of a^u as a Exp. u. This last notation is the one used by A. Cayley in his article "Function" in the *Encyclopædia Britannica*, 9th edition, 1879. He writes e^u as exp. u. This notation has found wide acceptance in works on the theory of functions.

Certain novelties in logarithmic notation were suggested in Germany, but failed of adoption. As early as 1786 Abel Bürja[4] indicated the logarithmic inverse of $2^3 = 8$ by the notation $\frac{8}{=} = 3$. In 1811 Rothe[5] used the symbols 8?2, which were adopted in 1823 by Martin Ohm. In 1860 Köpp advocated a third notation which cannot be represented with ordinary type. His suggestion was opposed by F. A. Stadnička of Prag. The need of a new notation for logarithmation is not evident; the usual "$\log_2 8$" seems to satisfy ordinary needs.

CLASSIFICATION OF LOGARITHMIC SYSTEMS.

A curious classification of logarithmic systems of real positive numbers was based by Hessel, of the University of Marburg, upon the theorem that, if $a > \sqrt[e]{e}$, where $e = 2.718\cdots$ and $\sqrt[e]{e} = 1.4445\cdots$, then $a^x > x$ for all (real) values of x. In his first group of logarithmic systems each number is greater than its (tabular) logarithm, as must be true when the base $a > \sqrt[e]{e}$. In his second system the base $a = \sqrt[e]{e}$; here there is one number, namely e, which is

[1] *Op. cit.*, p. 255.
[2] *Cambr. Phil. Trans.*, Vol. XI, Pt. III, p. 450.
[3] *Messenger of Mathematics*, Vol. II, 1873, pp. 107–111.
[4] A. Bürja, *Algebrist*, Berlin und Libau, 1. Theil, p. 153.
[5] Drænert in Hoffmann's *Zeitschr. f. Math. u. Phys.*, Vol. 8, Leipzig, 1877, p. 265.
[6] Grunert's *Archiv*, Vol. 14, 1850, p. 97.

equal to its own logarithm, but all other numbers are smaller than their respective logarithms. In his third group of systems, in which each base $a < \sqrt[e]{e}$, some of the numbers are larger, others are smaller than their respective logarithms, while for one number there is an equality between the two.

A general and more fundamental classification was given in 1892 by Irving Stringham[1] of the University of California. It was modified somewhat by M.W. Haskell and Stringham,[2] and formulated by Stringham in his *Uniplanar Algebra*, San Francisco, 1893. In the Wessel-Argand diagram, complex numbers are represented by vectors drawn from the origin to points on some logarithmic spiral; the logarithms of these numbers are represented by vectors from the origin to points on a straight line. The base is the particular vector drawn to the spiral which corresponds to the vector 1 drawn to the line; the modulus is, in general, a complex number. Logarithmic systems are classified as *gonic* and *agonic* systems, according as the modulus is a complex number or a real number.

LOGARITHMS AS DIRECT FUNCTIONS.

That it is possible to approach logarithms, not from the viewpoint of inverse functions, but from the standpoint of direct functions, was recognized early in the eighteenth century. John Bernoulli I, in his correspondence with Leibniz, started from the differential equation $dx/x = -dx/-x$ and thence deduced $\log x = \log(-x)$. This operation presented the logarithmic function as an integral, but labored under the defect of not making the integral definite. Clear ideas on this subject were entertained by Gauss who in a letter of Dec. 18, 1811, addressed to Bessel,[3] set forth the significance of $\int_1^x dx/x$ in the Wessel-Argand plane as an infinitely many-valued function of a complex variable. This view has been presented in works on the theory of functions.[4]

In 1869 an article was published by Cayley, *On logarithms of imaginary quantities*[5] in which the logarithm is defined as a definite integral and the general power a^b is defined as the inverse of the logarithmic function. Much the same course was pursued in 1902 by J. W. Bradshaw.[6] A radical change is advocated by Felix Klein. After discussing the difficulties encountered in defining a logarithm by the equation $a^b = c$, he goes so far as to recommend that, for purposes of school room instruction, the logarithm be defined as an integral and be connected with the asymptotic areas of the equilateral hyperbola.[7] He declares that the proper source for the introduction of new functions is the quadrature of familiar curves. This program is elaborated more fully in an article by C. Frenzel.[8]

[1] *American Jour. of Math.*, Vol. XIV, 1892, pp. 187–194.
[2] *Bulletin of the N. Y. Math. Society*, Vol. II, 1893, pp. 164–170.
[3] Gauss, *Werke*, Bd. VIII, p. 90.
[4] H. Durège, *Theorie der Funktionen einer compl. veränd. Grösse*, Leipzig, 3. Aufl., 1882, Absch. V.
[5] Cayley, *Coll. Math. Papers*, Vol. VI, p. 14.
[6] *Annals of Math.*, 2d S., Vol. 4, 1902–3; pp. 51–62. See also W. F. Osgood, *Funktionentheorie*, Leipzig u. Berlin, 1907, pp. 487–498.
[7] F. Klein, *Elementarmathematik vom höheren Standpunkte aus.*, I, Leipzig, 1908, p. 332.
[8] *Zeitschrift f. math. u. naturwiss. Unterricht*, 44. Jahrg., 1913, pp. 1–13.

A purely analytical theory of the logarithmic function, divorced from geometry, was developed in 1891 by Ch. Méray of Dijon.[1] He considered the logarithmic function as the result of integration and arrived at the exponential function by solving for u the equation $\log u = x$.

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ADDITIONS AND CORRECTIONS.

Page 9, line 23; G. Enestrom points out that Halley's definition was given earlier by N. Mercator in his *Logarithmo-Technia*, 1768, thus: *Est enim logarithmus nihil aliud, quam numerus ratiuncularum contentarum in ratione, quam absolutus quisque ad unitatem obtinet* See also our footnote (2), page 9.

Page 13, line 2 up; *for* 1644 *read* 1649.

Page 44, equation (1) should read $i\varphi = \log(i \sin\varphi + \cos\varphi)$. Likewise, each ψ on this page should be replaced by φ. Line 16 up, *for* $e^{+v\sqrt{-1}}$ and $e^{-v\sqrt{-1}}$ *read* $e^{+v\sqrt{-1}}$ and $e^{-v\sqrt{-1}}$.

Page 45, line 1; *for* 1902 *read* 1903. Footnote (1), *for* 1902 *read* 1903.

Page 83, line 8 up; G. Enestrom has pointed out that Euler gave a value of i^i as early as 1746, in a letter to Goldbach. The letter is found in P. H. Fuss, *Correspondance mathématique et physique de quelques célèbres géomètres du XVIII^ème siècle*, tome I, 1843, p. 383. The passage is as follows: *Letztens habe gefunden, dass diese expressio* $(\sqrt{-1})^{\sqrt{-1}}$ *einem valorem realem habe, welcher in fractionibus decimalibus* = 0,2078795763, *welches mir merkwürdig zu seyn scheinet.* Euler makes no reference here to the infinitely many values of i^i.

Page 149, line 3 up; *for* w plane *read* z plane. Line 2 up; *for* values of w *read* values of z.

Page 150, line 6; *for* conection *read* connection. Line 15; *for* z-plane *read* w-plane, *for* w-surface *read* z-surface.

Page 179, lines 15–19 were intended as part of footnote (2).

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[1] *Théorie analyt. de log. népérien et de la fonct. exponent.*, par M. Ch. Méray, Paris, 1891.

THE SIGNIFICANCE OF WEIERSTRASS'S THEOREM.

By E. R. HEDRICK, University of Missouri.

1. Introduction. Certain theorems that are commonly presented in mathematical instruction appear at first sight to be of theoretical importance only. Actual application of some such theorems seems very far fetched, and it is often not sought for because the possibility of its realization seems slight.

Among such theorems is that of Weierstrass on polynomial approximations to a continuous function:*

Given a function $f(x)$ continuous in the interval $a \leqq x \leqq b$, and a positive number ϵ, then a polynomial $P(x)$ exists, such that

$$|f(x) - P(x)| < \epsilon, \qquad a \leqq x \leqq b.$$

It is the purpose of this paper to show how this theorem can be clarified to students, and how the contrast between the approximations that it expresses and those given by Taylor's series can be illuminated.

2. Contrast with Taylor Approximating Polynomials. The characteristic property of the successive polynomial approximations

$$T_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \tag{1}$$

that are given by a Taylor series, is that *the coefficient a_i of a given power x^i is the same for every approximation past the first in which it occurs.*

The polynomials $P(x)$ of Weierstrass's theorem do not in general have this property, even when $f(x)$ is analytic. If $f(x)$ is not analytic, it is impossible that they should have that property.

To illustrate the manner in which polynomial approximations may arise that do not have the Taylor property, let us consider, for example, the amount of stretching in a steel wire due to variable tension. Hooke's law is expressed by the formula

$$y = y_0(1 + kx), \tag{2}$$

where y_0 is the original length of the wire, and y is its length under a tension x. The constant k is determined by experiment upon a given wire.

It is known that Hooke's law is only approximately correct. To take into account minor variation not expressed by that law, we may assume a formula of the type

$$y = y_0(1 + k_1x + k_2x^2), \tag{3}$$

and determine the constants k_1 and k_2 by a series of experiments. If this be done, *the coefficient k_1 of x in* (3) *may or may not, logically, coincide with the value of k in* (2). If it does, it manifests the characteristic Taylor property; if not, the polynomial approximations (2) and (3) are typical rather of the Weierstrass behavior. As a matter of fact, as we shall see below, k_1 is *not* equal to k.

* See, for example, Goursat-Hedrick, *Mathematical Analysis*, Vol. I, p. 422.

For still greater accuracy, or for theoretical purposes, we might insert still higher powers of x in the assumed formula for y:

$$y = y_0(1 + k_1^{(n)}x + k_2^{(n)}x^2 + \cdots + k_n^{(n)}x^n), \tag{4}$$

where $k_i^{(n)}$ denotes the coefficient of x^i in the nth one of these approximations. The question that arises is whether $k_i^{(n)}$ is or is not independent of n; if it is, the polynomials have the Taylor property mentioned above; if not, the coefficient of the same power of x varies from one approximation to another in the Weierstrass manner. We proceed to show that the latter alternative is the usual one.

3. Computation of Coefficients by Least Squares. In such instances as that mentioned in article 2, the coefficients k_i are usually computed by the method of least squares. If the relation between two observed quantities x and y is of the form (4), it is usual to select the values of the coefficients $k_i^{(n)}$ which render the sum of the squares of the differences between the two sides of (4) a minimum, the sum being extended over all the sets of observed values. This is equivalent to solving for $k_i^{(n)}$ the following set of linear equations:

$$\left\{\begin{array}{l} k_1^{(n)}\Sigma x_j^2 + k_2^{(n)}\Sigma x_j^3 + \cdots + k_n^{(n)}\Sigma x_j^{n+1} = \Sigma x_j\left(\dfrac{y_j - y_0}{y_0}\right), \\ k_1^{(n)}\Sigma x_j^3 + k_2^{(n)}\Sigma x_j^4 + \cdots + k_n^{(n)}\Sigma x_j^{n+2} = \Sigma x_j^2\left(\dfrac{y_j - y_0}{y_0}\right), \\ \quad\cdot\qquad\quad\cdot\qquad\quad\cdot\qquad\quad\cdot \\ \quad\cdot\qquad\quad\cdot\qquad\quad\cdot\qquad\quad\cdot \\ \quad\cdot\qquad\quad\cdot\qquad\quad\cdot\qquad\quad\cdot \\ k_1^{(n)}\Sigma x_j^{n+1} + k_2^{(n)}\Sigma x_j^{n+2} + \cdots + k_n^{(n)}\Sigma x_j^{2n} = \Sigma x_j^n\left(\dfrac{y_j - y_0}{y_0}\right), \end{array}\right. \tag{5}$$

where x_j and y_j are a pair of corresponding observed values of x and y, and where the summations are extended over all the pairs of observed values.

It is evident from these formulas that the approximation of next higher degree will give a set of coefficients $k_i^{(n+1)}$ that are in general very different from $k_i^{(n)}$, since the inclusion of a term in x^{n+1} in (4) results in increasing the number of equations in (5), as well as in adding a term to each left-hand side. It follows that *the successive approximations given by the theory of least squares have the characteristic Weierstrass property, i. e., the coefficient $k_i^{(n)}$ of x^i depends on n.*

4. Failure of Taylor Series. It is well known that many continuous functions, and even many functions that possess all their derivatives, are not expansible in Taylor series. Thus the function e^{-1/x^2} possesses all its derivatives for every value of x; nevertheless it is not expansible in Taylor series. A function that fails to have a derivative—of no matter how high an order—for a given value of x, is not expansible in Taylor series about that point. It is a result of Weierstrass's theorem that such approximations as (4) above, are not affected by these failures of Taylor's series.

We have seen in Art. 3 that the approximations of the form (4) ought not to be considered as the first terms of a Taylor series. It now becomes evident

that it is not even desirable to do so if it were allowable. For there are serious limitations upon the applicability of Taylor's (infinite) series, whereas *any continuous function* may be treated by Weierstrass's theorem. Doubtless most scientific phenomena are continuous in the range of experimentation; they certainly are so to within the errors of observation.

It follows that the frequent occurence in science of polynomial approximations of the type (4) constitutes an illustration of Weierstrass's theorem rather than of Taylor's theorem. Quite contrary to the spirit of apologetic hesitation that is sometimes manifested by experimenters in using what are popularly known as "the first terms of a Taylor series" to represent an observed quantity, Weierstrass's theorem constitutes a complete justification for the use of formulas of the type (4) in any scientific experiment.

ON THE IMPOSSIBILITY OF CERTAIN DIOPHANTINE EQUATIONS AND SYSTEMS OF EQUATIONS.*

By R. D. CARMICHAEL, Indiana University.

INTRODUCTION.

The purpose of this paper is to give simple proofs of several theorems concerning Diophantine equations of specified form and to derive certain interesting consequences of them. All of these proofs are supposed to be novel, in whole or in part; and the results in VI, VII and VIII are believed to be new.

In §1 it is shown that the Diophantine system $q^2 + n^2 = m^2$, $m^2 + n^2 = p^2$ is without a solution (in positive integers). Several of the remaining theorems of the paper are proved by means of this one. In §2 it is shown that no numerical right triangle has a square area. In §3 the impossibility (in positive integers) of the equation $p^4 - q^4 = \alpha^2$ is proved, and several consequences of this fact are obtained. In §4 I show that the equations $m^4 - 4n^4 = \pm t^2$ are both impossible (in positive integers), and derive various consequences.

Incidentally, several proofs are given of the impossibility of the Fermat equation $x^4 + y^4 = z^4$.

The principal arguments in §§1 and 4 are made by means of Fermat's famous method of "infinite descent."

§1. *Impossibility of the simultaneous equations* $q^2 + n^2 = m^2$, $m^2 + n^2 = p^2$.

In this section we shall prove the following theorem:

THEOREM I. *There do not exist integers* m, n, p, q, *all different from zero, such that*

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* Presented to the American Mathematical Society, March 21, 1913.

(1) $$q^2 + n^2 = m^2, \quad m^2 + n^2 = p^2.$$

It is obvious that an equivalent theorem is the following:

THEOREM II. *There do not exist integers m, n, p, q, all different from zero, such that*

(2) $$p^2 + q^2 = 2m^2, \quad p^2 - q^2 = 2n^2.$$

The method of proof is to assume the existence of integers satisfying equations (1) and (2) and to show that we are thus led to a contradiction. The argument we give is an illustration of Fermat's famous method of "infinite descent."

If any two of the numbers p, q, m, n have a common prime factor t, it follows at once from (1) and (2) that all four of them have this factor. For, consider an equation in (1) or in (2) in which these two numbers occur; this equation contains a third number, and it is readily seen that this third number is divisible by t. Then from one of the equations containing the fourth number it follows that this fourth number is divisible by t. Now, let us divide each equation of system (2) by t^2; the resulting system is of the same form as (2). If any two numbers in this resulting system have a common prime factor t_1, we may divide through by t_1^2; and so on. Hence if a pair of simultaneous equations (2) exists then there exists a pair of equations of the same form in which no two of the numbers m, n, p, q have a common factor other than unity. Let this system of equations be

(3) $$p_1^2 + q_1^2 = 2m_1^2, \quad p_1^2 - q_1^2 = 2n_1^2.$$

We shall show that the assumption of a pair of equations (3) leads to a contradiction. Obviously, we may without loss of generality take p_1, q_1, m_1, n_1 to be all positive; and this we do.

From the first equation in (3) it follows that p_1 and q_1 are both even or both odd; and, since they are relatively prime, it follows that they are both odd. Evidently, $p_1 > q_1$. Then we may write

$$p_1 = q_1 + 2\alpha,$$

where α is a positive integer. If we substitute this value of p_1 in the first equation of (3), the result may readily be put in the form

(4) $$(q_1 + \alpha)^2 + \alpha^2 = m_1^2.$$

Since q_1 and m_1 have no common prime factor it is easy to see from this equation that α is prime to both q_1 and m_1, and hence that no two of the numbers $q_1 + \alpha$, α, m_1 have a common factor.

Now it is well known that if a, b, c are positive integers no two of which have a common prime factor, while

$$a^2 + b^2 = c^2,$$

then there exist relatively prime integers r and s, $r > s$, such that

$$c = r^2 + s^2, \quad a = 2rs, \quad b = r^2 - s^2$$

or

$$c = r^2 + s^2, \quad a = r^2 - s^2, \quad b = 2rs.$$

Hence from (4) we see that we may write

$$q_1 + \alpha = 2rs, \quad \alpha = r^2 - s^2 \tag{5}$$

or

$$q_1 + \alpha = r^2 - s^2, \quad \alpha = 2rs. \tag{6}$$

In either case we have

$$p_1^2 - q_1^2 = (p_1 - q_1)(p_1 + q_1) = 2\alpha \cdot 2(q_1 + \alpha) = 8rs(r^2 - s^2).$$

If we substitute in the second equation of (3) and divide by 2 we have

$$4rs(r^2 - s^2) = n_1^2.$$

From this equation and the fact that r and s are relatively prime it follows at once that r, s, $r^2 - s^2$ are all square numbers; say,

$$r = u^2, \quad s = v^2, \quad r^2 - s^2 = w^2.$$

Now $r - s$ and $r + s$ can have no common factor other than 1 or 2; hence from

$$w^2 = r^2 - s^2 = (r - s)(r + s) = (u^2 - v^2)(u^2 + v^2)$$

we see that either

$$u^2 + v^2 = 2w_1^2, \quad u^2 - v^2 = 2w_2^2 \tag{7}$$

or

$$u^2 + v^2 = w_1^2, \quad u^2 - v^2 = w_2^2.$$

And if it is the latter case which arises, then

$$w_1^2 + w_2^2 = 2u^2, \quad w_1^2 - w_2^2 = 2v^2. \tag{8}$$

Hence, assuming equations of the form (2) we are led either to equations (7) or to equations (8); that is, we are led to new equations of the form with which we started. Let us write the equations thus:

$$p_2^2 + q_2^2 = 2m_2^2, \quad p_2^2 - q_2^2 = 2n_2^2; \tag{9}$$

that is, system (9) is identical with that one of systems (7), (8) which actually arises.

Now from (5) and (6) and the relations $p_1 = q_1 + 2\alpha$, $r > s$, we see that

$$p_1 = 2rs + r^2 - s^2 > 2s^2 + r^2 - s^2 = r^2 + s^2 = u^4 + v^4.$$

Hence $u < p_1$. Also,

$$w_1^2 \lneqq w^2 \lneqq r + s < r^2 + s^2.$$

Hence $w_1 < p_1$. Since u and w_1 are both less than p_1 it follows that p_2 is less than p_1. Hence, obviously, $p_2 < |p|$. Moreover, it is clear that all the numbers p_2, q_2, m_2, n_2 are different from zero.

From these results we have the following conclusion: If we assume a system

of the form (2) we are led to a new system (9) of the same form; and in the new system $| p_2 |$ is less than $| p |$.

Now if we start with (9) and carry out a similar argument we shall be led to a new system

$$p_3^2 + q_3^2 = 2m_3^2, \quad p_3^2 - q_3^2 = 2n_3^2,$$

with the relation $| p_3 | < | p_2 |$; and so on *ad infinitum*. But, since there is only a finite number of positive integers less than the given integer $| p |$ this is impossible. We are thus led to a contradiction; whence we conclude at once the truth of II, and likewise of I.

§2. *The area of a numerical right triangle is never a square number.*

This famous theorem of Fermat is readily proved by means of the results in the preceding section. Let the sides and hypotenuse of a numerical right triangle be u, v and w, respectively. The area of this triangle is $\frac{1}{2}uv$. If we assume this to be a square number t^2 we should have the following simultaneous Diophantine equations

$$u^2 + v^2 = w^2, \quad uv = 2t^2. \tag{10}$$

We shall prove our theorem by showing that the assumption of such a system leads to a contradiction.

If any two of the numbers u, v, w have a common prime factor p then the remaining one also has this factor, as one sees readily from the first equation in (10). From the second equation in (10) it follows that t also has the same factor. Then if we put $u = pu_1$, $v = pv_1$, $w = pw_1$, $t = pt_1$, we have

$$u_1^2 + v_1^2 = w_1^2, \quad u_1 v_1 = 2t_1^2,$$

a system of the same form as (10). It is clear that we may start with this new system and proceed in the same manner as before, and so on, until we arrive at a system

$$\bar{u}^2 + \bar{v}^2 = \bar{w}^2, \quad \bar{u}\bar{v} = 2\bar{t}^2, \tag{11}$$

where $\bar{u}$, $\bar{v}$, $\bar{w}$ are prime each to each.

It is well known that the general solution of the first equation (11) may be written in one of the forms

$$\bar{u} = 2ab, \qquad \bar{v} = a^2 - b^2, \quad \bar{w} = a^2 + b^2;$$

$$\bar{u} = a^2 - b^2, \quad \bar{v} = 2ab, \qquad \bar{w} = a^2 + b^2.$$

Then from the second equation in (11) we have

$$t^2 = ab(a^2 - b^2) = ab(a - b)(a + b).$$

It is easy to see that no two of the numbers a, b, $a - b$, $a + b$ in the last member of this equation have a common factor; for, if so, $\bar{u}$ and $\bar{v}$ would have a common factor, contrary to hypothesis. Hence each of these four numbers is a square. That is, we have equations of the form

$$a = m^2, \quad b = n^2, \quad a + b = p^2, \quad a - b = q^2;$$

whence

$$m^2 - n^2 = q^2, \quad m^2 + n^2 = p^2.$$

But, according to theorem I, no such system of equations can exist.

Hence we have the following theorem:

THEOREM III. *The area of a numerical right triangle is never a square number.*

COROLLARY. *The equation* $\alpha^4 + 4\beta^4 = \gamma^2$ *is impossible in integers* α, β, γ, *all of which are different from zero.*

For, from $\alpha^4 + 4\beta^4 = \gamma^2$ it would follow that a right triangle whose sides and hypotenuse are respectively α^2, $2\beta^2$, γ has the square area $\alpha^2\beta^2$, which is impossible.

In the next section additional proofs will be given of this corollary and theorem.

§3. *Impossibility of the equation* $p^4 - q^4 = \alpha^2$. *Consequences.*

If there exists a set of integers p, q, α all different from zero, such that

$$p^4 - q^4 = \alpha^2, \tag{12}$$

it is easy to see that there exists such a set having also the further property that they are prime each to each. For, if any two of the numbers contain the prime factor t, the third contains this factor also and the equation may be divided through by t^4. Continuing this process one arrives finally at an equation of the form (12) in which p, q, α are prime each to each.

We shall now show that the assumption of such an equation (12) leads to a contradiction of theorem I or of theorem II. We have

$$(p^2 - q^2)(p^2 + q^2) = \alpha^2.$$

Since p and q are relatively prime the two factors of the first member of this equation evidently have the greatest common factor 1 or 2; for the greatest common factor is a divisor of $2q^2$, the difference of the numbers. Hence $p^2 - q^2$ and $p^2 + q^2$ are either both square numbers or both the double of square numbers. That is, we have a system of the form

$$p^2 - q^2 = m^2, \quad p^2 + q^2 = n^2,$$

or a system of the form

$$p^2 - q^2 = 2m^2, \quad p^2 + q^2 = 2n^2.$$

But, according to theorems I and II, each of these systems is impossible.

Thus we have the following theorem:

THEOREM IV. *There do not exist integers* p, q, α, *all different from zero, such that*

$$p^4 - q^4 = \alpha^2;$$

that is, the difference of two fourth power integers cannot be a square.

Second proof. This theorem may also be readily proved by means of theorem III. For, assuming (12) we have

$$(p^4 - q^4)p^2q^2 = p^2q^2\alpha^2. \tag{13}$$

But, obviously,

$$(2p^2q^2)^2 + (p^4 - q^4)^2 = (p^4 + q^4)^2. \tag{14}$$

Now, from (13) we see that the numerical right triangle determined by (14) has its area $p^2q^2(p^4 - q^4)$ equal to the square number $p^2q^2\alpha^2$. But this is impossible. Hence no equation of the form (12) exists.

COROLLARY. *The system* $p^2 - q^2 = km^2$, $p^2 + q^2 = kn^2$ *is impossible in integers* p, q, k, m, n, *all of which are different from zero.*

For, if we take the product of these two equations member by member we have as a result an equation which by theorem IV is impossible.

Since from $x^4 + y^4 = z^4$, we have $z^4 - x^4 = (y^2)^2$ the above theorem leads to the following:

THEOREM V. *The equation* $x^4 + y^4 = z^4$ *has no solution in integers all of which are different from zero.*

By means of theorem IV one may readily prove the corollary to theorem III. Let us assume an equation of the form

$$\alpha^4 + 4\beta^4 = \gamma^2.$$

If α and β have a common prime factor p, γ has also this factor, and we may divide the equation by p^4, the resulting equation being of the same form as the original one. This process may be repeated until we have an equation

$$\alpha_1{}^4 + 4\beta_1{}^4 = \gamma_1{}^2,$$

where α_1 and β_1 are relatively prime. It is then easy to see that γ_1 has no odd factor in common with either α_1 or β_1. Suppose that α_1 and γ_1 have the factor 2 in common. We may then divide the equation by 4, thus obtaining a new equation of the same form. We may continue this process until the α and the γ of the resulting equation are not both even. It is then obvious that both must be odd. Let this equation be

$$\bar{\alpha}^4 + 4\bar{\beta}^4 = \bar{\gamma}^2.$$

Then $\bar{\alpha}^2$, $2\bar{\beta}^2$ and $\bar{\gamma}$ are relatively prime. Then from the theory of numerical right triangles it follows that we have equations of the form

$$\bar{\beta}^2 = ab, \qquad \bar{\alpha}^2 = a^2 - b^2, \qquad \bar{\gamma} = a^2 + b^2;$$

where a and b are relatively prime positive integers. From the first of these equations we see that a and b are both square numbers. Then the second of the equations is of the form (12). Since no equation of this form exists we conclude the impossibility of the equation $\alpha^4 + 4\beta^4 = \gamma^2$; that is, we have the corollary of theorem III.

Having demonstrated this corollary without the use of theorem III we may now employ the corollary to prove III itself. Let the argument read as in the preceding case down to equations (11). From the second equation in (11) we see that $\bar{u}$ and $\bar{v}$ have this property: One of them is a square number, say α^2,

and the other is twice a square number, say $2\beta^2$. Then the first equation in (11) takes the form $\alpha^4 + 4\beta^4 = \bar{w}^2$. This equation being impossible we conclude the truth of theorem III.

§4. *Impossibility of the equations* $m^4 - 4n^4 = \pm t^2$. *Consequences.*

We shall assume the existence of one of the equations

$$m^4 - 4n^4 = \pm t^2, \tag{15}$$

in which m, n, t are all different from zero, and show that we are led to a contradiction. If any two of the numbers m, n, t have a common odd prime factor p then all three of them have this factor, and the equation may be divided through by p^4. The new equation thus obtained is of the same form as the original one. This process may be repeated until an equation

$$m_1^4 - 4n_1^4 = \pm t_1^2$$

is obtained in which no two of the numbers m_1, n_1, t_1 have a common odd prime factor.

If t_1 is even it is obvious that m_1 is also even, and therefore the above equation may be divided by 4; a result of the form

$$n_1^4 - 4m_2^4 = \mp t_2^2$$

is obtained. The process can be continued until an equation

$$m_3^4 - 4n_3^4 = \pm t_3^2$$

is obtained in which t_3 is an odd number. Then m_3 is also odd. If the second member of the above equation has the minus sign then we have

$$(m_3^2)^2 + t_3^2 = 4n_3^4;$$

that is, we have the sum of two odd squares divisible by 4, which is impossible since an odd square is congruent to 1 modulo 4. Hence we must have

$$4n_3^4 + t_3^2 = m_3^4. \tag{16}$$

Moreover, it is obvious that no two of the numbers n_3, t_3, m_3 have a common prime factor and that all of them are different from zero.

From (16) and the theory of numerical right triangles we have equations of the following form:

$$n_3^2 = rs, \quad t_3 = r^2 - s^2, \quad m_3^2 = r^2 + s^2,$$

where r and s are relatively prime positive integers. From the first of these equations it follows that r and s are squares; say, $r = \rho^2$, $s = \sigma^2$. Then from the last equation we have

$$\rho^4 + \sigma^4 = m_3^2.$$

It is easy to see that ρ, σ and m_3 are prime each to each.

For this equation we have the general solution

$$m_3 = r_1^2 + s_1^2, \quad \rho^2 = 2r_1s_1, \quad \sigma^2 = r_1^2 - s_1^2,$$

or the general solution

$$m_3 = r_1^2 + s_1^2, \qquad \rho^2 = r_1^2 - s_1^2, \qquad \sigma^2 = 2r_1s_1.$$

In either case we see that $2r_1s_1$ and $r_1^2 - s_1^2$ are both squares, while r_1 and s_1 are relatively prime and one of them is even. From the relation $r_1^2 - s_1^2 =$ square, it follows that r_1 is odd, since otherwise we should have the sum of two odd squares equal to the even square r_1^2, which is impossible. Hence s_1 is even. But $2r_1s_1 =$ square. Then, since r_1 is odd we have r_1 equal to a square. Then $2s_1$ is a square, or s_1 is twice a square. Then we may write $r_1 = \rho_1^2$, $s_1 = 2\sigma_1^2$. Hence we have an equation of the form

$$\rho_1^4 - 4\sigma_1^4 = w_1^2$$

since $r_1^2 - s_1^2$ is a square; that is, we have

$$4\sigma_1^4 + w_1^2 = \rho_1^4. \tag{17}$$

Now the last equation has been obtained solely from (16). Moreover it is obvious that no one of the numbers σ_1, w_1, ρ_1 is zero. Also, we have

$$n_3^2 = rs = \rho^2\sigma^2 - 2r_1s_1(r_1^2 - s_1^2) = 4\rho_1^2\sigma_1^2(\rho_1^4 - 4\sigma_1^4) = 4\rho_1^2\sigma_1^2w_1^2.$$

Hence $\sigma_1 < n_3$. Similarly, starting from (17) we should be led to an equation

$$4\sigma_2^4 + w_2^2 = \rho_2^4,$$

where $\sigma_2 < \sigma_1$; and so on indefinitely. But such a recursion is impossible.

We have been led to this contradiction by the initial assumption that an equation of the form (15) is possible. Hence we have the following theorem:

THEOREM VI. *Neither of the equations* $m^4 - 4n^4 = \pm t^2$ *is possible in integers* m, n, *and* t, *all of which are different from zero.*

COROLLARY. *The system of equations* $p^2 - 2q^2 = km^2$, $p^2 + 2q^2 = \pm kn^2$ *is not possible in integers* p, q, k, m, n, *all of which are different from zero.*

For, if we take the product of these equations member by member we have as a result an equation which by theorem VI is impossible.

From the corollary to theorem VI we have also the following interesting result:

THEOREM VII. *The area of a numerical right triangle is not twice a square number.*

For, if there exists a set of numbers u, v, w such that

$$u^2 + v^2 = w^2, \qquad uv = 4k^2 = t^2$$

then it is easy to see that

$$(u + v)^2 = w^2 + 2t^2,$$
$$(u - v)^2 = w^2 - 2t^2.$$

Since from the preceding corollary these equations are impossible, the truth of the theorem follows at once.

From theorems III and VII we have the following corollary:

COROLLARY. *The number expressing the area of a numerical right triangle has at least one odd prime factor entering to an odd power.*

Now a number of the form $rs(r^2 - s^2)$, where r and s are different positive integers, is the area of the numerical right triangle determined by the equation

$$(2rs)^2 + (r^2 - s^2)^2 = (r^2 + s^2)^2.$$

Put $r = \rho^2$ and $s = \sigma^2$. Then from the above corollary we see that $\rho^2\sigma^2(\rho^4 - \sigma^4)$ has some odd prime factor entering into it to an odd power. Since every prime factor of $\rho^2\sigma^2$ obviously enters into $\rho^2\sigma^2$ to an even power, we have the following result:

THEOREM VIII. *The number $\rho^4 - \sigma^4$, in which ρ and σ are different positive integers, has always an odd prime factor entering into it to an odd power.*

As immediate corollaries of this theorem we have theorems IV and V in §3.

By means of theorem VII we may also prove the following theorem:

THEOREM IX. *The equation $m^4 + n^4 = \alpha^2$ is impossible in integers all of which are different from zero.*

For, if such an equation exists we have a right triangle $(m^2)^2 + (n^2)^2 = \alpha^2$ whose area $\frac{1}{2}m^2n^2$ is twice a square number, which is impossible.

Another proof of theorem V by means of theorem IX is immediate. Theorems IV and IX, taken together, may be stated in the form:

THEOREM X. *In a numerical right triangle $a^2 + b^2 = c^2$, not more than one of the numbers a, b, c is a square.*

PROBLEMS AND QUESTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

ALGEBRA.

392. Proposed by H. PRIME, Boston, Mass.

The floor and ceiling of a room h feet high are parallel and horizontal. In the middle of the ceiling is a vertical circular tube w feet in diameter extending upward from the ceiling. From the room the longest possible inflexible rod is put up the tube. When the rod is in contact with the opposite sides of the tube and the floor, what is the expression for the tangent of the acute angle made by the rod and the floor in terms of h and w, considering the rod as an air line. To be solved without the calculus. [From *The Maine Farmers' Almanac*, 1913.]

393. Proposed by H. E. TREFETHEN, Waterville, Maine.

Expand $\log [(\sin x)/x]$ in terms of x.

394. Proposed by E. B. ESCOTT, University of Michigan.

Solve the equation $\sin^2 x \sin^2 2x = 5/16$.

395. Proposed by V. M. SPUNAR, Chicago, Ill.

Solve the system of equations $x_1^2x_2 = a_1$; $x_2^2x_3 = a_2$; $x_3^2x_4 = a_3$; $\cdots$ $x_n^2x_1 = a_n$.

GEOMETRY.

420. Proposed by C. N. SCHMALL, New York City.

Four spheres are described so that each touches a face of a given triangular pyramid and the other three faces produced. If the radii of the escribed spheres be r_1, r_2, r_3, r_4, and r be the radius of the inscribed sphere, show that

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}=\frac{2}{r}.$$

421. Proposed by R. P. BUSSEY, University of Iowa.

Assuming the details of the proof of the existence of a sphere inscribed in a tetrahedron as usual in the texts, give an intuitional proof that there are in general eight spheres each touching the four faces, but for the regular tetrahedron only five. How many special types are there?

422. Proposed by J. SCHEFFER, Hagerstown, Md.

Construct strictly by use of elementary plane geometry, a triangle having given the product of two sides, the median to the third side, and the difference of the angles adjacent to the third side.

423. Proposed by C. N. SCHMALL, New York City.

The sum of the squares of the distances of a point from n fixed points is constant. Show that the locus of the point is a circle.

CALCULUS.

342. Proposed by CLARIBEL KENDALL, University of Colorado.

Referring to Poincaré's *Science and Hypothesis*, page 65, show that the path between two points, in Poincaré's ideal world, requiring the least number of steps of beings such as exist on earth, is the arc of a circle cutting the boundary of the Poincaré world orthogonally.

343. Proposed by C. N. SCHMALL, New York City.

Show that the envelope of the system of circles, whose radii are the ordinates of an ellipse, is a concentric ellipse having the same minor axis as the given ellipse. Does this ellipse touch all the circles of the system?

344. Proposed by B. F. FINKEL, Drury College.

Solve the differential equation,

$$\frac{d^2y}{dx^2}(a^2-x^2-y^2)=\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}.$$

337. *Correction.* For $\int_0^1(\cos 2\pi qx+\cos 2\pi bx)dx$ read $\int_0^1|\cos 2\pi ax+\cos 2\pi bx|\,dx$.

MECHANICS.

273. Proposed by A. M. HARDING, University of Arkansas.

A spherical shell of mass m explodes when moving with negligible velocity at a height of h feet above the ground. The shell is divided into very small particles, each of which moves, after the explosion, away from the center of the shell with a velocity v, and ultimately falls to the ground. Find the total mass of the fragments which will be found per unit area at any specified distance from the point vertically underneath the shell.

274. Proposed by W. W. LANDIS, Dickinson College.

A dam backs up the water for two miles. If the dam is raised 18 inches, will the water two miles up the stream be raised 18 inches, more or less?

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

190. Proposed by L. E. DICKSON, University of Chicago.

Find two numbers a and b each of two digits such that, when the product ab is found by the usual method, the two partial products to be added are exactly the same as the partial products in getting the product ba. Is there a similar pair of numbers of three digits?

191. Proposed by E. B. ESCOTT, University of Michigan.

Find triangles whose sides are integers and one of whose angles is 60°.

192. Proposed by CHARLES MACAULEY, Chicago, Ill.

Combinations containing an even number of letters are formed of the letters a, b, c, d, etc. It is required to place the letters in two columns, so that half the letters in every combination are placed in one column and the other letters of the combination in the other column, and so that all the a's are placed in the same column; all the b's in the same column; all the c's in the same column, etc.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

383. Proposed by E. B. ESCOTT, University of Michigan.

Find accurately to 6 decimals $1.000000854^{1233451}$.

SOLUTION BY THE PROPOSER.

Using natural logarithms,

$$\log_e (1 + x) = x - \frac{x^2}{2} + \cdots.$$

Then

$$\log_e 1.000000854 = .000000854 -, + \cdots,$$

and the natural logarithm of

$$1.000000854^{1233451} \text{ is } 1.096066686.$$

Hence the common logarithm is $1.096066686 \times .434294482$, or .4760157. And the number required is 2.992373.

384. Proposed by H. C. FEEMSTER, York, Nebraska.

A man addressed n envelopes and wrote n checks in payment of n bills. Show that the number of ways of enclosing within each envelope one bill and one check in such a manner that in no instance all the enclosures shall be correct is

$$n!\left\{n! - \frac{(n-1)!}{1!} + \frac{(n-2)!}{2!} \cdots + (-1)^n \frac{0!}{n!}\right\}, \text{ where } 0! = 1.$$

SOLUTION BY THE PROPOSER.

The number of ways in which the enclosures in one letter may be correct is ${}_nC_1((n-1)!)^2$ except ${}_nC_2((n-2)!)^2$ which have been counted twice, except also ${}_nC_3((n-3)!)^2$ of these latter, which have been counted twice, etc. So the number of ways in which the enclosures in one letter may be correct is

$${}_nC_1((n-1)!)^2 - {}_nC_2((n-2)!)^2 + {}_nC_3((n-3)!)^2 - \cdots + (-1)^{n-1}{}_nC_{n-2}(2!)^2 + (-1)^n {}_nC_{n-1}(1!)^2 + (-1)^{n+1} {}_nC_n(0!)^2.$$

Hence the number of ways in which both enclosures in a single envelope may be incorrect is:

$$(n!)^2 - {}_nC_1((n-1)!)^2 + {}_nC_2((n-2)!)^2 - {}_nC_3((n-3)!)^2 + \cdots + (-1)^{n-2} {}_nC_{n-2}(2!)^2 + (-1)^{n-1}{}_nC_{n-1}(1!)^2 + (-1)^n{}_nC_n\, 0!$$

or

$$(n!)^2 - \frac{n((n-1)!)^2}{1!} + \frac{n(n-1)}{2!}((n-2)!)^2 - \frac{n(n-1)(n-2)}{3!}((n-3)!)^2$$

$$+ \cdots (-1)^{n-2}\frac{n(n-1)}{2!}(2!)^2 + (-1)^{n-1}n(1!)^2 + \cdots + (-1)^n 1\cdot 0!,$$

where $0!$ is defined as 1. Or it may be written:

$$n!\left[n! - \frac{(n-1)!}{1!} + \frac{(n-2)!}{2!} - \frac{(n-3)!}{3!} + \cdots + (-1)^{n-2}\frac{2!}{(n-2)!} + (-1)\frac{1!}{(n-1)!} + (-1)^n\frac{0!}{n!}\right].$$

GEOMETRY.

412. Proposed by S. LEFSCHETZ, University of Nebraska.

To inscribe in a given circle an isosceles triangle in which the base plus the altitude is equal to a given length.

SOLUTION BY A. M. HARDING, Univ. of Arkansas.

From A, any point on the circle, draw a diameter and produce it to P making $AP = l$, the given length. At A draw AQ tangent to the circle and equal to $\frac{1}{2}l$. Join P and Q. Assume that the line PQ cuts the circle in two points B and B'.

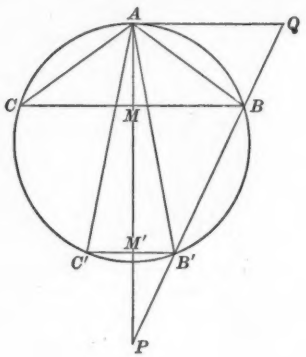

Fig. 1.

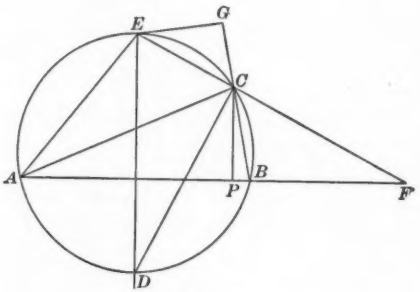

Fig. 2.

From B and B' draw chords perpendicular to AP cutting AP at M and M' and cutting the circle at C and C' respectively. Then ABC and $AB'C'$ are the isosceles triangles required. See Fig. 1.

Proof.

$$\frac{PM}{MB} = \frac{PA}{AQ} = \frac{2}{1}.$$

Hence

$$PM = 2MB = CB.$$

Then $AM + CB = AM + PM = l$. Likewise, $AM' + C'B' = l$. There will be two solutions, one solution, or no solution according as PQ cuts the circle, is tangent to the circle, or lies outside of the circle. In other words the problem is possible only when $2r < l < (1 + \sqrt{5})r$.

Also solved analytically, by H. C. FEEMSTER.

413. Proposed by DAVID R. KELLEY, New York City.

To construct a triangle, given the base, the vertical angle, and the ratio of the altitude from the vertex on the base to the difference of the other two sides.

SOLUTION BY THE PROPOSER.

Let ABC be the required triangle. Then since side AB and $\angle C$ are known, we can construct the circumcircle of the triangle. Let the bisectors of the interior angle C and the exterior angle C meet the circle in D and E, respectively. Then D and E are fixed points, and ED is the diameter of the circle, and AE is fixed. Draw EG perpendicular to BC produced. Then $CG = \frac{1}{2}(AC - BC)$. Hence, if CP be the altitude from C on AB, CG/CP is also known, since by hypothesis $(AC - CB)/CP$ is the given ratio. Again, since in the right triangle ECG, $\angle GCE$ is known, therefore GC/EC is known, and therefore, by above, EC/CP is known. Now let EC meet AB in F. Then triangles DEC and CFP are similar. Hence, $EC/CP = DE/CF$. But EC/CP is given. Hence DE/CF is also known, and therefore, since DE is the diameter of the circumcircle of triangle ABC, CF can be found. Also $EC \cdot EF = \overline{EA}^2$. Hence, since CF and EA are known, EC can be constructed and hence point C can be found. See Fig. 2.

Also solved by H. C. FEEMSTER, C. N. SCHMALL, and A. M. HARDING.

414. Proposed by H. C. FEEMSTER, York, Neb.

To construct the cyclic quadrilateral, having given its four sides.

SOLUTION BY THE PROPOSER.

Construction.—Let the lengths of the given sides be a, b, c, d. Draw $AD = d$, and produce it to E so that $DE : c = a : b$. Now describe a circle which is the locus of a point P such that $EP : PA = c : b$ (circle of Apollonius). Describe also a circle with the center D and radius c. Let C be one point of intersection of these circles. Then on AC as a base construct a triangle ABC similar to EDC. Then $ABCD$ is the quadrilateral required. See Fig. 3.

Proof.—By construction $AD = d$, $DC = c$. Now since the angles ABC and EDC are equal, therefore $\angle ABC$ and $\angle ADC$ are supplementary and the figure $ABCD$ is cyclic. It now remains to prove that $AB = a$, and $BC = b$.

By construction, $ED : c = a : b$, or $ED : DC = a : b$. Also by construction $EC : CA = c : b$, or $EC : DC = CA : b$. Hence

$$ED : DC : EC = a : b : CA. \tag{1}$$

But by similar triangles, we have

$$ED : DC : EC = AB : BC : CA. \tag{2}$$

Then by (1) and (2), $AB = a$, $BC = b$.

There is evidently but one solution; for if the other point of intersection of the circles be taken instead of C, the resulting quadrilateral will be the same in size but drawn on the opposite side of AD since the intersections are symmetrical with regard to AD.

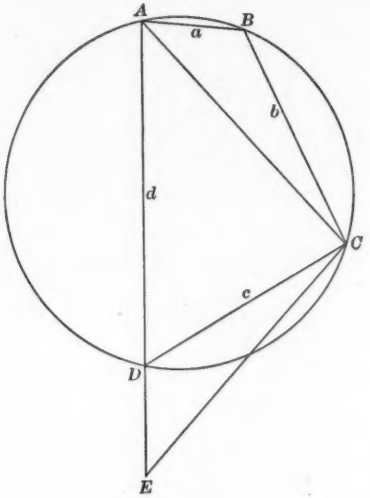

Fig. 3.

Note.—In this construction it is assumed that the order of the sides a, b, c, d, is given. If this order is not given it may be shown that three cyclic quadrilaterals can be drawn having these sides.

MECHANICS.

265. Proposed by A. H. HOLMES, Brunsw'ck, Maine.

A gun is mounted in a fort at height h above the sea, and a similar gun is mounted on a ship. Show that there is a region of area $4\pi rh$ within which the ship is within range of the fort while the fort is out of range of the ship, r being the maximum range of either gun on a horizontal plane through it.

SOLUTION BY A. M. HARDING, University of Arkansas.

If a projectile is fired at an angle α its range R on a level plane at a distance h below the point of projection is given by the equation

$$2v^2(h + R\tan\alpha) = gR^2\sec^2\alpha.$$

That is

$$R = \frac{v}{g}\cos\alpha\,[v\sin\alpha + \sqrt{(v^2 + 2gh) - v^2\cos^2\alpha}].$$

From the value of $dR/d\alpha$, set equal to zero, we have as the condition for a maximum range,

$$\sin^2\alpha = \frac{v^2}{2(v^2 + gh)} \qquad \text{and} \qquad \cos^2\alpha = \frac{v^2 + 2gh}{2(v^2 + gh)}.$$

Hence $R = \sqrt{r^2 + 2hr}$, where r is the maximum horizontal range. This is the maximum range for a gun in the fort. Likewise the maximum range for a gun on the ship is

$$R_1 = \sqrt{r^2 - 2hr}.$$

Therefore the area required is

$$\pi R^2 - \pi R_1^2 = 4\pi hr.$$

Also solved by MRS. H. E. TREFETHEN and H. C. FEEMSTER.

NUMBER THEORY.

185. Proposed by R. D. CARMICHAEL, Bloomington, Ind.

Obtain the complete solution of the equation $\phi(p^\alpha) = \phi(q^\beta)$ where ϕ denotes Euler's ϕ-function, p and q are unknown primes, and α and β are unknown integers.

SOLUTION BY L. C. MATHEWSON.

From the theory of numbers[1] we have

$$\begin{aligned} \phi(p^\alpha) &= p^\alpha\left(1 - \frac{1}{p}\right) = p^{\alpha-1}(p-1); \\ \phi(q^\beta) &= q^\beta\left(1 - \frac{1}{q}\right) = q^{\beta-1}(q-1). \end{aligned} \tag{1}$$

Obviously the given equation is always satisfied by $p = q$, if $\alpha = \beta$. We, accordingly, consider the case where the primes are not equal, or $p \neq q$.

From (1) we have

$$p^{\alpha-1}(p-1) = q^{\beta-1}(q-1). \tag{2}$$

Since $p^{\alpha-1}$ and $q^{\beta-1}$ can have no common factors, we see that (2) gives

$$\begin{aligned} p^{\alpha-1} &= q - 1 \\ q^{\beta-1} &= p - 1 \end{aligned}. \tag{3}$$

It is sufficient to consider only the first of these equations, since the second is of the same form and leads to no new general results or solutions. Now $\alpha = 1$ or $\alpha > 1$. From (3) written in the form $p^{\alpha-1} + 1 = q$, we see that if $\alpha = 1$, $q = 2$, the even prime: if $\alpha > 1$, p cannot be an odd prime (for this would make q composite), hence p can be only 2. Since we are now excluding $p = q$, and because from (3) we have seen that the results will be interchangeable, and since we now see that at least one of the primes here must be the even prime, we consider p odd and q even. We have then to discuss $\phi(p^\alpha) = \phi(q^\beta)$, where p is an odd prime, $\alpha = 1$, $q = 2$; i. e., $\phi(p) = \phi(2^\beta)$.

We have simply $p - 1 = 2^{\beta-1}$, or $p = 2^{\beta-1} + 1$. Here p is an odd prime of the form $2^m + 1$. It is easily seen that m cannot be an odd prime nor an integral multiple of an odd prime (for in these cases since the binomial form itself is the

[1] See Bachmann's Neuere Zahlentheorie, p. 24, or any elementary text on the theory of numbers.

sum of the same odd powers of two integers, it is factorable); m can thus be only an integral power of 2; as, 1, 2, 4, 8, ,$\cdots$. This means that $\beta = 2^n + 1$, where n must be such a positive integer that $p = 2^{2^n} + 1$ is a prime.

Summarizing, we have for solutions of the given equation $\phi(p^\alpha) = \phi(q^\beta)$:

Case (1), $p = q$. Any prime, even or odd, is a root, if $\alpha = \beta$.

Case (2), $p \neq q$. One root is the even prime 2 with any exponent 2^n such that $2^{2^n} + 1$ is a prime, this odd prime being the other root for this individual solution. In this case the exponent of the odd prime is unity, while that of the even prime is of the form 2^n, as just indicated.

It may be added that the general form for n such that $2^{2^n} + 1$ is always a prime is still undetermined. It is of interest to note that this form is one which Fermat conjectured was always a prime for positive integral values of n. While this is true for many values of n, as 0, 1, 2, 3, $\cdots$, the factors have been found for other values of n: thus $2^{2^n} + 1$ is composite for $n = 5, 6, 9, 11, 12, 18, 23, 36, 38$.[1] These numbers are too large to be handled readily (or at all) when expressed in ordinary Arabic notation. The French encyclopedia of mathematics (t. 1: 3, p. 51) states that $2^{2^{36}}$ is an integer of more than twenty billion digits. The two following interesting propositions have been proposed, but no published proofs seem to exist: All the numbers represented in the series $2 + 1$, $2^2 + 1$, $2^{2^2} + 1$, $2^{2^{2^2}} + 1$, $\cdots$ are primes. The second is due to Eisenstein: There is an infinity[2] of prime integers of the form $2^{2^n} + 1$.

186. Proposed by H. PRIME, Boston, Mass.

Show that $\dfrac{(n+1)(n+2)\cdots(2n-2)}{(n-1)!}$ is an integer for all values of n.

SOLUTION BY THOMAS E. MASON, Bloomington, Ind.

The expression in the problem is equivalent to (1) $(2n-2)!/n!(n-1)!$ and the expression (2) $(m+n)!/m!n!$ is an integer, since it is a coefficient in the binomial expansion for the exponent $m + n$.

Each prime factor of the denominator of (1) which is not a factor of n occurs in the denominator as many times as it occurs in $(n-1)!(n-1)!$, and therefore occurs at least as many times in the numerator, since by (2) $(2n-2)!/[(n-1)!(n-1)!]$ is an integer.

Any prime factor of n (different from 1) is not contained in $n - 1$, and therefore occurs in the denominator of (1) as many times as it occurs in $n!(n-2)!$. It occurs at least as many times in the numerator since by (2) $(2n-2)!/n!(n-2)!$ is an integer.

The expression in the problem is therefore an integer for all values of n since the numerator contains every prime factor of the denominator at least as many times as it occurs in the denominator.

Also solved by B. F. YANNEY and H. C. FEEMSTER.

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[1] Intermed. Math., 10 (1903), p. 158; 11 (1904), p. 79. Lucas, Théorie des Nombres, p. 51.

[2] Lucas, Théorie des Nombres, p. 355.

BOOK REVIEWS.

School Algebra. Book I. By GEORGE WENTWORTH and DAVID EUGENE SMITH. Ginn and Company, Boston, 1913. iv + 298 pages. $0.90.

This is the third of the Wentworth-Smith Series. The authors have prepared a two years course in algebra for high schools, to be issued in two volumes, of which this is the first. The plan of the work is similar to that of their Academic Algebra, reviewed in the June number of the MONTHLY, but the treatment of topics is somewhat more extended.

R. R. SHUMWAY.

Les Anaglyphes Géométriques. By H. VUIBERT. Librairie Vuibert, Paris, 1912. 32 pages. Francs 1.50.

Some sixty years ago Wilhelm Rollmann of Stralsund published a paper on the use of two-colored images with glasses of two colors (not necessarily complementary). In 1874, 1875 and 1876 L. Ducos du Hauron of Algiers published several articles on héliochromie. The latter investigator imposed one upon the other two stereoscopic views in complementary colors and gave to the picture thus formed the name *anaglyphe*. By looking at this picture with the right eye through a red medium, with the left eye through a green, a marvelous stereoscopic effect is secured, the figure seeming to float in space.

Profiting by the discoveries of Ducos du Hauron, M. Richard exhibited at the international congress of mathematicians in Cambridge last year some forty anaglyphs which were much admired. More recently MM. Richard and Vuibert have published this little book of 32 pages containing thirty specimen anaglyphs illustrative of solid and descriptive geometry, crystallography and physics. Each copy contains a pair of red and green "glasses" mounted in card board. It is the purpose of the authors to issue a series of albums of anaglyphs some of which will be of special interest to teachers in secondary schools.

W. W. BEMAN.

Trigonometry. By ALFRED MONROE KENYON and LOUIS INGOLD. Edited by EARLE RAYMOND HEDRICK. The Macmillan Company, New York, 1913. xi + 132 pages. *Logarithmic and Trigonometric Tables.* Prepared under the direction of EARLE RAYMOND HEDRICK. The Macmillan Company, New York, 1913. xvii + 124 pages. $1.35.

The Kenyon-Ingold Trigonometry is the second of the new series of mathematical books published by The Macmillan Company under the general editorship of Professor E. R. Hedrick, of the University of Missouri. Like the Davis Calculus, the first of the new series, it departs somewhat from the traditional course. The first marked departure is that the solution of triangles is attacked immediately and is completely solved in the first three chapters (47 pages), the addition formulas, trigonometric equations and identities, and all references to angles greater than 180° being left out because they are extraneous. The trig-

onometric functions are first defined for acute angles and are used in solving right triangles. Then they are defined for obtuse angles and are used in solving oblique triangles. The student is taught the fundamental principles of solution by means of the *law of sines* and the *law of cosines*. The table of powers and roots at the back of the book makes the latter law practicable for computation. Then he is taught the special logarithmic methods that make use of the *law of tangents* and the *law of tangents of the half-angles*. Geometric proofs are given for these two laws. The proofs usually given are not available because they involve the addition formulas and the formulas for double and half angles which come later in this book.

The first sentence of the preface sounds the keynote of the book. "In trigonometry, as elsewhere, a motive for the study of each topic is necessary to secure the effective attention of the student." The motive so far in the book is the solution of triangles. In most of the current texts, this is made to serve as the only motive. The solution of triangles is begun early and finished late in the course. After the student has learned to solve right triangles, he begins the study of what is called goniometry or trigonometric analysis. He learns about the functions of angles of all sizes, and about the addition formulas and the formulas to which they lead. He knows that the goal toward which he is striving is the solution of oblique triangles; and after he has reached that goal, he wonders why he has had to solve so many equations and prove so many formulas and identities that seem to him unnecessary. The solution of triangles is one of the most interesting problems of elementary mathematics, and it will be more interesting and more satisfactory to the student if it is presented in a straightforward manner with everything extraneous left out, as it is done in this book.

The other parts of trigonometry, which are more important for analytic geometry and the calculus, are taken up in chapters 4, 5, 6, and 7. Directed lines and angles, radian measure, functions of any angle, graphs, the addition formulas, equations and identities, and the inverse functions follow in order. The composition and resolution of forces is made an introduction to the study of angles greater than 180° and is used to illustrate the meaning of the addition formulas. Large angles are used in problems on rotation and angular speed, and the radian is shown to be useful in problems on rotation and mensuration. The applications of large angles include forces, velocities, simple harmonic motion, and vibrations.

So far in the book a motive has been found for each topic. There has been a minimum of abstract trigonometric manipulation. The next chapter deals with identities, equations, and inverse functions. It is unusually complete and well written, especially the part on trigonometric equations. It contains no practical applications.

The authors have used rectangular coordinates from the very beginning of the book. Many teachers believe that plotting with polar coordinates should be a part of every course in trigonometry because it gives the student facility in handling the trigonometric functions, especially those of angles greater than 180°,

and because it is a good preparation for analytic geometry. The authors have not put this topic in their book. Perhaps they thought it would have to be dragged in without any motive that the student could see.

The last chapter in the book (25 pages) is devoted to Spherical Trigonometry.

The logarithmic and trigonometric tables prepared under the direction of E. R. Hedrick are unusually complete. They are five-place tables and include tables of powers, roots, and reciprocals, and natural logarithms. Brief four-place tables are added for use in problems that do not demand extreme accuracy. The introduction explains the slide rule and gives a graphic representation of interpolation.

Answers to problems are not given.

W. H. BUSSEY.

NOTES AND NEWS.

The August number of *The Popular Science Monthly* contains an article entitled "Bernoulli's Principle and its application to explain the curving of a base ball."

Mr. E. J. Moulton has been promoted to an assistant professorship in mathematics at Northwestern University. He has just taken the doctor's degree at the summer convocation at the University of Chicago.

John Wiley and Sons have recently published "The Theory of Relativity," by Professor Robert D. Carmichael, of the University of Indiana, as No. 12 of the series of mathematical monographs edited by Mansfield Merriman and Robert S. Woodward. The price is $1.00.

Mr. H. R. Kingston, graduate student at the University of Chicago, has been appointed to an instructorship in mathematics at the University of Manitoba, Winnipeg, Canada.

The Macmillan Company has published an *edition de luxe* of "The Davis Calculus," printed on India paper and bound in flexible leather. The thickness of the book is one half that of the regular edition. The price is $2.40. The price of the regular edition is $2.00.

DR. W. KILLING has for the second time been awarded the Lobachevski prize of the Physico-Mathematical Society of Kasan.

Miss Martha McDonald, graduate student at the University of Chicago, is to be in charge of the department of mathematics at Oxford College for Women, Oxford, Ohio.

"Tables of the Exponential Function" computed to twenty places, compiled by C. E. VAN ORSTAND, were published in the June number of the *Journal of the Washington Academy of Sciences.*

At the University of Chicago H. E. SLAUGHT and G. A. BLISS have been promoted to full professorships in mathematics.

M. PIERRE BOUTROUX will be on the faculty of Princeton University this fall as a professor of mathematics. He is closely related to the Poincaré family.

At Stanford University W. A. MANNING has been promoted to an associate professorship in mathematics and H. F. BLICHFELDT to a full professorship.

Mr. H. N. WRIGHT, who received the doctor's degree at the University of California last May, has taken charge of mathematics in the University Extension work of the University of California.

Miss JOSEPHINE E. BURNS, who received the doctor's degree at the University of Illinois last June, has been appointed instructor in mathematics in the same institution in place of Dr. R. M. Winger, who accepted an assistant professorship in the University of Oregon.

On the twenty-fifth anniversary of President Slocum's assumption of office at Colorado College, appropriate celebration ceremonies were held at the June Commencement. On this occasion the trustees bestowed upon Dean FLORIAN CAJORI the honorary degree of Doctor of Laws. A few days later the University of Wisconsin conferred the degree Sc.D. upon Professor Cajori.

The July number of *The Monist* contains the following articles of interest to mathematicians: "The New Mechanics," by the late Henri Poincaré; "The Principle of Relativity as a Phase in the Development of Science," by Paul Carus; and "Robert Hooke as a Precursor of Newton," by Philip E. B. Jourdain.

The Eastern Association of Physics Teachers held its sixty-fourth meeting in Boston on March 29, 1913. An interesting item on the program was the report of a committee on new books. In this way important books are brought before teachers for discussion. Possibly this may be a good suggestion for meetings of mathematics associations.

The attendance of advanced students in mathematics at the University of Chicago summer quarter has been unusually large. All courses have been full but notably those in Theory of Functions and Integral Equations offered by Professor Bolza have been overflowing. The weekly meetings of the Mathematical Club have drawn large numbers. The programs have been devoted alternately to the discussion of pedagogical questions and the reading of technical papers of a research character.

During the last Summer Session of the University of California the High School Teachers of California held various sessions. Two of these were devoted to the consideration of mathematical questions. The following papers were read during these two sessions: "Mathematical troubles of the freshman," by Professor G. A. MILLER; "A broader program for advanced mathematics in secondary schools," by Dr. H. W. STAGER; "Course in plane surveying," by Professor F. S. FOOTE; Arithmetic and the introductory high school," by Miss T. BROOKMAN; "Arithmetic and bookkeeping of everyday life," by Mr. F. M. POWELL; "The irreducible minimum in high school mathematics," by Mr. C. M. TITUS.

On June 19 a statue of LORD KELVIN, subscribed for by the citizens of Belfast, was unveiled in that city by Sir Joseph Larmor. Belfast is the birthplace of Lord Kelvin. Another statue of Lord Kelvin, erected by the contributions of his fellow citizens in Glasgow and the West of Scotland, has been placed in position by the side of the new Kelvin Avenue in Glasgow and will be unveiled on October 8 next. A window in Westminster Abbey, in memory of Lord Kelvin, was unveiled on July 15.

Weekly popular lectures on mathematics were given during the summer session of the University of Wisconsin as follows: "The history of Greek geometry," by Professor L. W. DOWLING; "Soap films and their mathematical theory, with demonstrations," by Professor ARNOLD DRESDEN; "Zeno's paradoxes of the flying arrow and of Achilles and the tortoise, an historical treatment," by Dean FLORIAN CAJORI of Colorado College; "On the teaching of arithmetic," a conference conducted by Professor W. W. HART.

The May number of *L'Enseignement Mathématique*, the official organ of the international commission on mathematical teaching, contains an interesting popular article by GINO LORIA of the University of Genoa on "Excentricités et mystères des nombres." The March number of the same journal contains an article by A. N. WHITEHEAD of University College, London, on "The Principles of Mathematics in relation to elementary teaching," which discusses some vital topics such as are suggested in the following quotation: "There is an idea, widely prevalent, that it is possible to teach mathematics of a relatively advanced type—such as differential calculus, for instance—in a way useful to physicists and engineers without any attention to its logic or its theory. This seems to me to be a profound mistake. . . . I fully admit that the proper way to start such a subject as the differential calculus is to plunge quickly into the use of the notation in a few absurdly simple cases, with a crude explanation of the idea of rates of increase. . . . The habit of logical precision with its necessary concentration of thought upon abstract ideas is not wholly possible in the initial stages of learning. It is the ideal at which the teacher should aim."

The final installment of Professor Cajori's *History of the Logarithmic and Exponential Concepts* is found in this issue, together with a table of contents and some additions and corrections. This series of articles is not to be reprinted separately and can be obtained only as an integral part of Volume XX of the MONTHLY. The interest in historical studies will be continued in the October issue with the first installment of *Number Systems of the North American Indians*, by W. C. Eells of Whitworth College, Tacoma, Washington. These studies are based upon a critical examination of about 300 languages and dialects. The linguistic diversity of the North American Indians is one of the most remarkable features of world ethnology, and Professor Eells discusses their number words and numer systems in a highly instructive and interesting manner.

The annual report by *Science* concerning "Doctorates conferred by American Universities" was printed in the issue of August 22. This report shows in tabular

form the average number of doctorates of philosophy and of science conferred by each of 44 universities during the ten year period, 1898–1907, the actual numbers for each year from 1908 to 1913, and the total numbers for the sixteen year period, 1898–1913. Previous to 1908 Chicago had been in the lead; since that date Columbia has forged rapidly ahead. During the sixteen years twelve universities have conferred more than one hundred doctorates each. These are in order: Columbia, 702; Chicago, 648; Harvard, 588; Yale, 522; Johns Hopkins, 475; Pennsylvania, 406; Cornell, 374; Wisconsin, 206; Clark, 159; New York, 149; Michigan, 125; and Boston, 104. Twelve others have conferred from thirty to eighty each; and eleven others, from ten to eighteen each. The grand total is 5237 in all departments.

A similar tabulation is given for doctorates conferred in the science departments alone. Here, only eight universities have conferred more than 100 doctorates each, and Chicago is decidedly in the lead. They are: Chicago, 333; Johns Hopkins, 283; Columbia, 281; Cornell, 255; Harvard, 235; Yale, 234; Pennsylvania, 161; and Clark, 145. The grand total is 2541 in all science departments.

A third table shows the numbers for various departments during the same periods. The totals for these are: Chemistry, 682; Physics, 311; Zoology, 286; Psychology, 275; Mathematics, 248; Botany, 240; English, 192; History, 152; Geology, 151; Economics, 125; Philosophy, 121. Fifteen other departments show a total of 25 to 97 each; six others a total of 12 to 20 each, the remaining departments showing less than ten each.

The number of doctorates in mathematics averaged 12.1 for the years 1898–1907, and 21.1 for the years 1908–1913, with a grand total of 248 for the 16 years. For 1913 there were 20 doctorates in mathematics distributed as follows: Harvard, 4; Columbia and Johns Hopkins, 3 each; Boston, Michigan, and Yale, 2 each; Chicago, Illinois, Pennsylvania, and Princeton, 1 each.

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MATHEMATICAL TROUBLES OF THE FRESHMAN[1]

By G. A. MILLER, University of Illinois.

The fact that there is generally a great gap between high-school mathematics and that of the freshman courses in our colleges and universities accounts for a considerable part of the troubles of the first year students in the higher institutions of learning. While it may not afford us any comfort to know that similar troubles exist in other countries, it is evidently desirable to acquaint ourselves with the extent of these troubles before we endeavor to explain their source or to devise means to mitigate them.

Conditions in Germany.—In the preface of his interesting book entitled *Die Erziehung der Anschauung*[2] Professor H. E. Timerding observes, in reference to conditions in Germany, that a deep gap divides the whole system of education. On the one side are the higher institutions of learning while on the other are the elementary schools. Everything is different on these two sides,—the objects, the methods, the views, the text-books; they are two completely separated worlds. The one has its center in scientific research, the other in practical education, and this separation is so complete that the results of scientific research come to the elementary teachers only through one or two intermediaries and in a very misleading form; while on the other hand, the university teachers pass by all pedagogic questions with great disdain.

Conditions in America.—Our American university student has perhaps one advantage over his German brother as a result of the fact that this one deep gap in the German system is generally replaced by two somewhat less serious ones in our system of mathematical education. One of these gaps is met on entering the freshman courses while the other is reached a few years later in the first courses of a really graduate nature. As the latter of these is encountered by a comparatively small number, whose deep interest inspires renewed courage, the former presents by far the more serious question to those interested in the

[1] Read before the Mathematics Section of the California High School Teachers Association, July 8, 1913.

[2] Published by B. G. Teubner in 1912.